Growth of cuspate spits

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ABSTRACT


The present work concerns cuspate spits, slightly symmetric geomorphic features growing along the shoreline in shallow waters. We develop a new working out for the dynamics of cuspate spits. Our development lies on classical paradigms such as a conservation law for the shoreface scale, and an explicit formula for the alongshore sediment transport. We derive a non-linear diffusion equation and a fully explicit solution for the growth of cuspate spits. From this general expression, we found interesting applications to quantify shoreline dynamics in the presence of cuspate spits. In particular, we point out a simple method for the datation of a cuspate spit given a limited number of input parameters. Furthermore, we develop a method to quantify the mean alongshore diffusivity along a shoreline perturbed by well-defined cuspate spits of known sizes. Finally, we introduce a formal relationship between the geometric characteristics (amplitude, length) of cuspate spits, which recall the self-similarity of these geomorphic features.

ADDITIONAL INDEX WORDS: nearshore, sand spit, Pelnard-Considère, non-linear diffusion equation

INTRODUCTION

A wide range of large-scale long-standing geomorphic features occur in shallow water environments, from tens of metres of water depth to the shoreline, either in the open sea or in continental settings. Ripples, megaripples, dunes and sandwaves develop in rhythmic or isolated patterns metre to kilometre long (Bruun, 1954; Bakker, 1968; Lonsdale and Malfait, 1974; McBride and Moslow, 1991; Reynaud et al., 1999; Lykousis, 2001; Todd, 2005; Raynal et al., 2009; Bouchette et al., 2010; Raynal et al., 2010). Sandbanks are a part of this family of bedforms, and include members such as mega-dunes, bars and ridges (Dyer and Huntley, 1999). Some sandbanks, termed shoreface-connected ridges and headland-associated banks, correspond to features that develop seaward from high points connected to the coast (McBride and Moslow, 1991; Dronkers, 2005). They are propagating down-drift, and they usually extend down to deep waters. Obviously, these local shoreline perturbations are associated to accumulation of sand.

Zenkovich (1959) first described cuspate spits (Fig. 1) as a limited category of shore-connected features that result from symmetrical wind/wave forcings and/or peculiar initial shore configuration (Bird, 1994; Coco and Murray, 2007). Asthon et al. (2001) and Asthon and Murray (2006) probed that cuspate spits, flying spits and other shoreline features derive from instabilities inherent in the relationship between alongshore sediment transport and local shoreline orientation. They exhibited a comprehensive weakly non-linear theory for cuspate and spit dynamics, and gave a striking numerical solutions to the problem.

The present work focuses on cuspate spits, also termed foreland spits, cuspate foreland or v-notches (Gilbert, 1885; Gulliver, 1896; Fisher, 1955; Zenkovitch, 1959), which are slightly symmetric shoreline-connected features that grow along the shoreline of shallow water environments. Basically, cuspates belong to the class of self-similar pattern i.e. as the time proceeds, shoreline varies while remains always geometrically similar to itself. From this point, we develop a new working out for the dynamics of cuspate forelands. We derive a non linear diffusion equation and an explicit solution for the dynamics of foreland spits. The final objective of this paper is to use the model developed with our approach to quantify mean growth velocity of cuspate spits, to contribute to the determination of their age or the mean longshore diffusivity at their origin. The paper also aims at providing additional ideas on the underlying physics of cuspate spits.

First, we recall the mechanical context driving the edification of cuspate spits, specifying what has been discussed already in the literature, and what we promote here. Then we present the main steps for the development and the proof of our mathematical model. Last, we adapt our cuspate model to various simple applied circumstances and we initiate a more formal discussion on the physics at the origin of cuspate spits.
THE NON-LINEAR PELNARD-CONSIDÈRE EQUATION

In this work, we make the assumption that seabottom and shoreline changes driven by strict cross-shore dynamics smooth and counterbalance over time. We consider that the consequence for the net change in the shoreline position over years is weak (Ruessink and Terwindt, 2000; Marino-Tapia et al., 2007). Indeed, at a long time scale, mean cross-shore profile is assumed to be at equilibrium (Hanson and Kraus, 1989; Dean, 1991), i.e. net cross-shore transport equals zero. The significant contribution to the long term shoreline change is thus from longshore dynamics (Allen, 1981; Aagaard and Greenwood, 1995). This assumption is at the origin of the formalism proposed here for the edification of cuspat spits.

Having this in mind, a basic mass balance equation states that the volume of sand required to move a profile cross shore is the shift of shoreline times the height of the active profile. Let be \( y = S(x, t) \) the equation of the shoreline position in a fixed \((x, y)\) coordinate system with the \( x \)-axis oriented alongshore, the \( y \)-axis oriented offshore and \( t \) the time (Fig. 2A). \( S(x, t) \) satisfies:

\[
\frac{dS}{dt} + \frac{1}{h_0 + B} \frac{dQ_L}{dx} = 0
\]  

(1)

where \( h_0 \) is the closure water depth (seaward which no significant transport occurs), \( B \) is the active berm height, \( h_0 + B \) is the height of the active profile (Fig. 2B). The total amount of sediment transported alongshore \( Q_L \) is related to the alongshore flux of energy available for the nearshore per unit length along the shoreline (Inman and Bagnold, 1963):

\[
Q_L(x) = \frac{KF_L(x)}{(\rho_s - \rho) g (1 - p)}
\]  

(2)

where \( \rho_s \) and \( \rho \) are densities of sediment and water respectively, \( p \) is the porosity, \( g \) is the acceleration of gravity. The dimensionless parameter \( K \) is an empiric constant. The energy flux to the beach \( F_L \) is defined by:

\[
F_L = C g \varepsilon_0 \cos(\delta_0 - \theta) \sin(\delta_0 - \theta)
\]  

(3)

where \( \cos(\delta_0 - \theta) \) is the ratio of incoming energy that goes from the closure water depth through the nearshore to the shoreline. In other words, it is the ratio of energy between two infinitely (dL) close wave rays that acts on an infinitely small dx shoreline segment (Fig. 2A). Forth, \( \sin(\delta_0 - \theta) \) is the longshore contribution of the total incoming energy. At the closure water depth \( h_0 \), the incoming energy is classically defined with the expression derived from linear wave theory:

\[
\varepsilon_0 = \frac{H_0}{8} \rho g H_0
\]

with \( H_0 \) the wave height. This energy propagates at the group velocity \( C_g \). This velocity must be calculated at the point where the energy flows into the active domain, that is at the closure water depth \( h_0 \). In this case, the linear wave theory provides the simple formulation:

\[
C_g = C_{g0} = \frac{g}{4 \pi} T_0
\]
The longshore transport rate \( Q_L \) is thus:

\[
Q_L = 2C_L \cos \left( \delta_0 - \theta \right) \sin \left( \delta_0 - \theta \right)
\]

with:

\[
C_L = \frac{K \rho H^2 T_0}{64 \pi (\rho_s - \rho) (1 - p)}
\]

Several formulations for the alongshore transport rate were successively derived from Eq. (3) (e.g. Komar and Inman, 1970; Komar, 1971; Bailard, 1984). Reviews and compared analyses of alongshore transport formulae were also performed (Bayram et al., 2001). Here, the sediment transport is stricly controled by \( F \), the longshore portion of flux of energy \( \varepsilon_0 \) per shoreline unit length.

No matter what type of wave transformation occurs in the nearshore. The only significant information is the fact that \( \delta_0 = 0 \) varies along the shoreline and that the energy that flows in the nearshore upward the shoreline depends upon this. This point of view is quite different from that chosen by other authors (for more explanations, see Ashton et al., 2001).

The combination of Eqs (4) and (1) is a model of long-term shoreline changes \( S(x, t) \) under mean wave forcings and mean sediment features. Until now, several strategies have been tested to find solutions for this kind of problem. First, Pelnard-Considère (1956) (and a significant subsequent literature) had the idea to linearize the problem so that a single linear diffusion equation describes the planform evolution of \( S(x, y) \). More recently, Ashton et al. (2001) and Asthon and Murray (2006) solved the problem numerically, with an expression of \( Q \) similar (but not equal) to that of Dean and Dalmryple (2002), and introduced a diffusion coefficient that depends on \( \theta \) (and may be thus negative). Asthon et al. (2001) focussed on rythmic foreland spits. Another striking idea was to introduce some non-linearity with an "expansion of the flow and the bottom perturbations in a truncated series of eigenfunctions of the linear problem" (Calvete et al., 2002; Falqués et al., 2008) which is not discussed here. The latter works concerned more specifically rythmics shoreline patterns like beach cusps. These works argued that shoreline connected features including foreland spits originate in instabilities. We provide a new solution of the problem.

The angle \( \theta \) between the local normal to the shoreline and the y-axis (Fig. 2C) satisfies:

\[
\sin \theta = \frac{\sqrt{1 + (dS/dx)^2}}{1 + (dS/dx)^2}
\]

\[
\cos \theta = \frac{1}{\sqrt{1 + (dS/dx)^2}}
\]

Eq. (4) can be rewritten in the following Eq (8):

\[
Q_L = C_L \left[ \sin 2 \delta_0 \left( \cos^2 \theta - \sin^2 \theta \right) - 2 \cos \delta_0 \sin \theta \cos \theta \right]
\]

Developing Eqs (6), (7) in Taylor series until order two in \( \partial S/\partial x \), and combining Eqs (1) and (8) results in

\[
\frac{dS}{dt} = G_0 \cos \delta_0 \frac{d^2 S}{dx^2} + 2 G_0 \sin \delta_0 \frac{dS}{dx} \frac{d^2 S}{dx^2}
\]

This is a nonlinear diffusion equation. When waves are directed along the x-axis (alongshore wind/wave forcings) sin \( 2 \delta_0 \) is zero and Eq. (9) reduces to a classical diffusion equation (Pelnard-Considère, 1956) with \( G_0 \) the longshore diffusivity. Another way to obtain Pelnard-Considère is to linearize Eq. (9). For this reason, we could name Eq. (9) the non-linear Pelnard-Considère equation.

In such a formulation, \( G_0 \) is given by:

\[
G_0 = \frac{C_L}{h_0 + B}
\]

And one will notice that

\[
G_0 = G \left( H_0, T_0, \delta_0, \rho_s, \rho_s, p, h_0 + B \right)
\]

which means that \( G_0 \) is a function of wave properties, sediment properties and basic geometrical informations.
DERIVATION OF THE CUSPATE EQUATION

From Eq. (11), we know that \( G_0 \) depends upon most of the “environmental” variables of the problem, i.e. those relative to the geometrical context and the forcings. As long term dynamics is mostly driven by mean values averaged to the historical/geological time scale, we can consider that \( H_0, T_0, p, \rho, \rho_0 \) are constant through time or vary very slowly. In the same manner, \( h_0 + B \) may not change as seabottom is always to the equilibrium (Short, 1999, p. 45, Fig 3).

Let us consider the following particular scenario in the frame \((0, x, y)\) (Fig. 2A). At \( t = 0 \), we have a non perturbed shoreline for \( x \in [-\infty, +\infty] \). At time \( t_0 \), a \((x, y)\)-symmetric and positive defined perturbation appears that develop on both sides the origin \( O \) and extend in \([-x_1, +x_1]\), being zero beyond. The building of such a cuspate spit supposes that the alongshore sediment transport results from two main dominant forcings varying close enough to \( \delta_0 = \pm \pi/4 \). Under these conditions Eq. (11) splits in two equations with solutions \( S_R \) (for \( \delta_0 = +\pi/4 \)) and \( S_L \) (for \( \delta_0 = -\pi/4 \)) satisfying:

\[
\frac{dS_{R/L}}{dt} = 2G_0 \sin(\pm \pi/2) \frac{d^2 S_{R/L}}{dx^2}
\]

As one can consider that the two forcings compete through time, we can substitute the real system represented by the two Eqs. (12) by a model based on a distributed solution satisfying:

\[
\frac{dS}{dt} = \begin{cases} 
0 & \text{for } x \leq -x_f \\
2G_0 \frac{d^2 S_R}{dx^2} & \text{for } -x_f < x \leq 0 \\
-2G_0 \frac{d^2 S_L}{dx^2} & \text{for } 0 \leq x \leq x_f \\
0 & \text{for } x \geq x_f
\end{cases}
\]

We already recalled that cuspate spits are self-similar patterns. Thus it is obvious to take into account the \((x,t)\) dependence of \( S \) through a self-similar variable \( \xi = \frac{x}{t^{1/3}} \) so that:

\[
S(x, t) \Leftrightarrow S(\xi, \tau) \text{ with } \xi = \frac{x}{t^{1/3}}
\]

Applying this variable substitution to the operators \( d/dx \) and \( d/dt \), we derive a new writing of Eq. (13):

\[
2G_0 S_R, \xi = \frac{1}{3} \xi = 0, \xi \in [-\xi_0, 0] \\
2G_0 S_L, \xi = \frac{1}{3} \xi = 0, \xi \in [0, \xi_0]
\]

Integrating twice, we obtain another expression with 4 distinct constants to be determined by the geometrical behavior of the cuspate spit. We impose to \( S \) to be continuous and positive defined at \( \xi = 0 \). In addition, we impose discontinuity of the derivative of \( S \) at \( \xi = 0 \). And we impose to \( S \) to be zero at the points \( \xi_\pm \) where \( S_\pm = 0 \). We obtain a set of equations with a single unknown parameter \( a \).

Going back to the original coordinates \((x, y)\), we get a new equation.

This equation is an exact solution of the problem developed in Eq. (9) adapted to the growth of any cuspate spit. Figure 3 displays some examples of plots of the expression \( S(x, y) \) derived here at various arbitrary times and for various values of the control parameters. Each curve could be cuspate spits like those in Fig. 1. The expression of the solution in the original coordinates is given by the following Eq. (16):

\[
\frac{dS}{dt} = \begin{cases} 
0 & x \leq -\sqrt{6at} \tau^{1/3} \\
\frac{1}{2G_0} \left[ \frac{2}{3} a \sqrt{6at} + \frac{ax}{t^{1/3}} - \frac{x^3}{18t} \right] & -\sqrt{6at} \tau^{1/3} \leq x \leq 0 \\
\frac{1}{2G_0} \left[ \frac{2}{3} a \sqrt{6at} - \frac{ax}{t^{1/3}} + \frac{x^3}{18t} \right] & 0 \leq x \leq \sqrt{6at} \tau^{1/3} \\
0 & x \geq \sqrt{6at} \tau^{1/3}
\end{cases}
\]

At this stage, the model must be developed further more. Indeed, unlike the longshore diffusivity \( G_0 \) which traduces the ability of the system to transport sediment alongshore, the parameter \( a \) has no clear physical meaning, as it simply results from an integration process. The plots in the Figure 3 are consistent with highly symmetric geomorphic features but, at this stage, we have no way to use the model for applications.
USING THE CUSPATE MODEL

From Eq. (27), the length $\lambda(t)$ of the foreland spit is:

$$\lambda(t) = 2\sqrt[3]{6at^{17/3}}$$

(17)

For $x=0$, Eq. (16) results in:

$$S(0) = \sqrt{6} \lambda^{3/2} / (3G_0)$$

(18)

Making power three Eq. (28), and deleting a to the power of 3/2 from equations, we get the expression:

$$S(0) = \frac{\lambda}{2G_0^{3/2}}$$

(19)

If $t_p$ is the present time, let call

$$\Delta t = t_p - t_0$$

(20)

the time required to nucleate and to develop the cuspate spit from an initial moment $t_0$. For convenience, let rename $S(0)$ with $A_p$.

With this formalism, Eq. (19) can be rewritten:

$$\Delta T = \frac{1}{2G_0}\frac{\lambda^3}{72A_p}$$

(21)

where $A_p$ and $\lambda_p$ are respectively the amplitude and the alongshore length of the cuspate spit in the present day. These parameters can be easily measured on an aerial photograph or by satellite imagery (Fig. 4), following the simple protocol described below.

To quantify $A_p$ and $\lambda_p$ for a given cuspate, the following method is applied: a) draw a line at the tangent to the shoreline; it intersects the shoreline in two points. This defines a segment which length is the cuspate length $\lambda_p$; b) from the head of the cuspate spit, draw the perpendicular line to the segment defined in (a); the point at their intersection and the point at the head of the cuspate form a new segment which length is the amplitude $A_p$ of the cuspate spit.

Eq. (21) can be considered as a method for the datation of a cuspate spit knowing the mean alongshore diffusivity and the geometrical features of the cuspate in the present day. Alternatively, the cuspate model can be thought as a tool to quantify the mean alongshore diffusivity $G_0$ having informations on the geometry of the cuspate spit and its age. Indeed, let consider that the present day geometry of a cuspate is known, as well as its age $\Delta t$ (e.g.: dated by its occurrence in an artificial continental water body dammed at a given period; or directly dated by any well-adapted datation method). These informations can be robustly retrieved from field works by geologists. From Eq. (21), it reads directly:

$$2G_0 = \frac{\lambda_p^3}{72A_p \Delta T}$$

(22)

The calculation of such a mean alongshore diffusivity from parameters expressed at the geological/historical time scale is interesting for the classification of the paleo-system concerned with respect to well-known existing littoral systems (such as described in Dean & Dalrymple, 2002). Another obvious application is to calculate the mean active profile (or the mean wave height or period) of the system at a given time knowing the present day geometry and all the other long-term mean forcings. Combining Eqs (5), (10) and (21), we have:

$$h_0 + B = \frac{9K \rho g H_0^2 T_0 A_p \Delta T}{2 \pi (\rho_s - \rho)(1 - p) \lambda_p^2}$$

(23)

The benefit of this last application could be debated for recent cuspates. Indeed, one will claim that there exists other ways to calculate an active profile directly from data in the field (e.g. considering the analysis of wave data at a buoy, and the subsequent extraction of a closure depth). Nevertheless, for ancient cuspate spits, it seems obvious that the calculation of $H_0^2 T_0$ directly from Eq. (23) would provide significant semi-quantitative restraints for the reconstruction of paleo-wave regimes. Indeed, the parameters $\rho_s$, $\rho$, $p$ could be determined by analysis of sand properties within the cuspate deposits. $\lambda_p$ and $A_p$ could be determined by direct surface observation (or seismic investigation if the cuspate is buried). $h_0$ could be determined by the identification of the location where wave ripples vanish, and $B$ by the identification of the position of the shoreline and the highest location concerned by wave impact.

A BRIEF AND MORE FORMAL DISCUSSION

The cuspate model offers the opportunity to calculate useful parameters for coastal engineering and geosciences. But applications under real conditions are far beyond the objective of this paper, although they could be presented in future works.

Nevertheless, the cuspate model highlights a more fundamental result. Indeed, in their numerical analysis of the growth of cuspate spit, Ashton & Murray (2006) suggested that there exists an approximated relationship between the age and the length of a cuspate. They wrote (Eq. (10) in their paper):

$$\Delta t \approx \left( \frac{K}{D_s} \frac{H_0^{12/5} T_0^{11/5}}{15} \right) x^2$$

(24)

where $\Delta t$ is our $\Delta T$ (the age of the cuspate), $\Delta x^2$ is $\lambda_p^2$ the cuspate length squared, and $D_s$, $K$, and others can be considered as constant and are ignored in the following. The authors derived this approximated relationship from the equation at the origin of their numerical model and from an analysis of numerical simulations.

In this paper, we demonstrate with Eq. (21) that a formal relationship between the age and the length of the cuspate is mathematically correct, and that this relationship is definitively controled by a power law as Ashton & Murray (2006) suggested. Although the term to the power is different (3 in our case, 2 in the case of Ashton & Murray), these two totally independent proofs give more confidence in the reality of such a relationship. The occurrence of a power law suggests also that more underlying physics remain to be analyzed.
Last, Ashton & Murray (2006) sustained the idea that, contrary to a traditional point of view, the wave angle in deep water strongly controls shoreline perturbations, through their so-called anti-diffusion Hocologic high wave angle instability. In this paper, we demonstrate that deep water wave features $H_{S}$ and $T_{S}$ drive non-linear shoreline instabilities in a simple way, with absolutely no need of what occurs to waves in the nearshore. This is also a formal confirmation of what was suggested by Ashton & Murray (2006). Forth, we demonstrate that there is no need to transform – by quite complex and obscure operations – shallow water variables into deep waters ones (Equations (1), (4) and (5) of Ashton & Murray, 2006) to get this result.

CONCLUSION

From what can be called the non-linear Pelnard-Considère equation, we develop an explicit formulation of the growth of a symmetrical cuspatte spit through time. This formulation can be applied (a) to the calculation of the age of a cuspatte spit, (b) to the determination of a mean alongshore diffusivity in the vicinity of a cuspatte spit, or (c) to the calculation of a paleo-active profile or informations on paleo-wave regimes after some simple geological field data. More substantially, the paper confirms the works of Ashton & Murray (2006) in the sense that it provides an alternative formal proof of what was suggested. In a near future, the cuspatte model will be more deeply explored, tentatively extended to other geomorphic features, and engineering/geological applications will be engaged to confirm its relevancy.

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